ESC103 Unit 20

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1 Inverse Matrices

In this unit, we will be working with square matrices (n×n).
Definition:
The matrix A is *invertible* if there exists another matrix, denoted as A^{-1}, such that:

$$A^{-1}A = I$$

and:

$$AA^{-1} = I$$

where I is the $n \times n$ identity matrix.

For example, any nonzero scalar has an inverse:

$$cc^{-1} = c\frac{1}{c} = 1$$

Important!:

$$\cancel{A^{-1} = \frac{1}{A}}$$

Notes 1: If matrix A is invertible, it cannot have more than one inverse. Let's suppose matrices B and C satisfy that

$$BA = I \qquad AC = I$$

then:

$$B(AC) = (BA)C$$

$$BI = IC$$
$$\therefore B = C$$
$$\therefore \text{B and C must be the same matrix}$$

Note 2: If matrix A is invertible, there is one, and only one solution to the system $A\overrightarrow{x} = \overrightarrow{b}$ and that solution is given by,

$$A\overrightarrow{x} = \overrightarrow{b}$$
$$A^{-1}(A\overrightarrow{x}) = A^{-1}\overrightarrow{b}$$
$$(A^{-1}A)\overrightarrow{x} = A^{-1}\overrightarrow{b}$$
$$I\overrightarrow{x} = A^{-1}\overrightarrow{b}$$
$$\therefore \overrightarrow{x} = A^{-1}\overrightarrow{b}$$

Note 3: If matrix A is invertible, then $A\overrightarrow{x} = \overrightarrow{0}$, can only have the trivial solution ($x = \overrightarrow{0}$).

$$A\overrightarrow{x} = \overrightarrow{0}$$
$$A^{-1}(A\overrightarrow{x}) = A^{-1}\overrightarrow{0}$$
$$(A^{-1}A)\overrightarrow{x} = \overrightarrow{0}$$
$$I\overrightarrow{x} = \overrightarrow{0}$$
$$\therefore \overrightarrow{x} = \overrightarrow{0}$$

$\therefore$ If there is a nonzero vector $\overrightarrow{x}$ such that $A\overrightarrow{x} = \overrightarrow{0}$, then matrix A is *not* invertible.

Note 4: A 2×2 matrix given by,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible *if and only if*:

$$ad - bc \neq 0$$

and its inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In linear algebra, $ad - bc$ is called *the determinant* of matrix A

Note 5: The product of two matrices AB has an inverse if and only if A and B are separately invertible (on their own) and the inverse of AB is $B^{-1}A^{-1}$.

If asked to prove that one matrix is the inverse of another, show that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

For the above example:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI)A^{-1} = A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B^{-1}(A^{-1}A)B = (B^{-1}I)B = B^{-1}B = I$$

By extension:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Example:
Start with elimination matrices $(n \times n)$. In unit 12, we defined an elimination matrix by applying *one* elimination step to the identity matrix.

It turns out that we can find the inverse of that same elimination matrix by applying the reverse of that elimination step to the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(R2 - 5R1)}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

To obtain E^{-1} as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{R2 + 5R1}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

Check: (Exercise):

$$E_1 E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check(Exercise:)

$$E_1^{-1} E_1 = I$$